REPORT OF COMMITTEE OF THE NEW ENGLAND ASSOCIATION OF TEACHERS OF MATHEMATICS ON SECONDARY SCHOOL MATHEMATICS, APRIL, 1916.

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I. INTRODUCTION.

The committee was appointed May 9, 1914, "to consider what action or attitude could wisely be taken by the Association with reference to more or less prevalent, and more or less responsible,

criticism of secondary school mathematics." Twelve meetings have been held.*

The announcement of the committee's appointment elicited an interesting letter from Dr. David Snedden—then Commissioner of Education in Massachusetts. This letter and a discussion of it were published in a preliminary report, December, 1914. The letter was particularly helpful to us in formulating in an authoritative way some of the current criticism which was in part the cause of our appointment, criticism ordinarily more apt to be circulated about teachers of mathematics than to be addressed to them.

Reference may also be made in this connection to an address to the Association by Superintendent Morrison of New Hampshire, December 5, 1914 (MATHEMATICS TEACHER ?).

Conferences and correspondence of interest have been had by the committee, or its representatives, with Professor David Eugene Smith, of Teachers College, Professor H. E. Hawkes, chairman of a similar committee of the Middle States Association, Professor P. H. Hanus, of Harvard University, Professor L. L. Conant, formerly of the Massachusetts State Board of Education, and many others.

Co-operative action has been proposed, or suggested, to similar organizations in other parts of the country—to the University Council of Massachusetts, to the American Mathematical Society, and to the recently organized Mathematical Association of America. Some of these proposals have been accepted and others may be expected to yield results of value in due time. Co-operation has also been initiated with the committee of this Association on Mathematics in the Pre-High-School Grades.

Besides this preliminary report, special informal reports of progress have been made at intervening meetings of the Association. Thus Mr. Evans has presented "In Defense of Algebra" (published in *School and Society*, March 18, 1916), and other members have reported on special topics named below. Dean Ferry has also discussed phases of our general problem at a

* It is worthy of note that while three of the eight members live at Worcester, South Hadley and Williamstown, respectively, every member has attended at least six meetings, while the average attendance has been more than six of the eight members. H. W. T.

meeting of the Middle States Association in New York, and at a meeting of the Worcester County Teachers' Association. It has seemed important that our campaign of education should be distributed to a considerable extent both in time and place.

II. Special Studies and Investigations.

Several special studies or investigations have been made by members of the committee, in particular: On the Preparation of Teachers of Mathematics in Massachusetts High Schools, by Dean Ferry; on Records of College Girls in Mathematics, and on Mathematics (or the lack of it) in Massachusetts Normal Schools, by Professor Smith; on the Value of Mathematical Study from the Standpoint of Modern Psychology, by Miss Pierce; on Requirements in Mathematics of Massachusetts High Schools, by Mr. Morse; on the Subject Matter of High School Algebra, by Mr. Evans, with the co-operation of Messrs. Galbraith and Meserve. Two of these special reports—on Preparation of Teachers, and on High School Requirements—are appended to the present report; that on the Value of Mathematics is expected to be published in *School Science and Mathematics*. The results of these special studies are incorporated to a greater or less extent in the body of this report.

The committee has also made much use of published investigations by the International Commission on the Teaching of Mathematics; the American Report, Committees 3 and 4, "Mathematics in the Public and Private Secondary Schools in the United States," by J. C. Brown. See also Appendix 3 on Literature.

III. Discussion.

Present Condition.—Algebra and geometry have long enjoyed a protected, or even monopolistic, position in the high-school curriculum. This position is now challenged: (*a*) by those who object to any prescription whatever; (*b*) by those who object to mathematics, discrediting the value of it from the standpoint of modern psychology and emphasizing the repugnance of students for it, and its limited future utility.

These critical or hostile views gain relative importance from the fact that the determination of educational policies and of

high-school curricula has fallen more or less into the hands of administrative officials entertaining such opinions. The continually increasing competition of newer subjects tends to weaken the position of the traditional subjects. Some teachers of mathematics in the high schools are inadequately prepared for their work, perhaps teaching it only incidentally. Conditions in normal and grammar schools are probably worse. There is unquestionably much room for improvement, and for more economical use of time in mathematical programs and textbooks.

Fundamental Issues.—The committee regards the following questions and considerations as fundamentally important:

1. Is there general public dissatisfaction with the present status of high-school mathematics? If so, to what extent is the criticism well founded and what is the remedy?

2. Is there sufficient reason either for making mathematics a more general requirement for high-school graduation, or for making it more completely optional than now?

3. Is there sufficient reason for asking the colleges to place less emphasis on mathematical requirements for admission?

4. Is the time devoted to mathematics in the schools excessive or deficient, in view of the results attained and the pressure for time?

5. Can better results be accomplished (*a*) by a modified program? (*b*) by a change in the character of textbooks and methods? (*c*) by better preparation of teachers?

Our conclusions may be summarized as follows: That dissatisfaction is not general; that the more radical criticism is not well-founded; that the monopolistic condition is no longer general; that college admission requirements in mathematics are not unreasonable; that most high school teachers in Massachusetts are reasonably well prepared.

On the other hand it will appear that there is need of improvement in the curriculum by eliminating matters too technical or of minor importance, substituting material of greater interest and less difficulty; need of improvement also in the quality of instruction particularly in the seventh and eighth grades.

Public Opinion.—The evidence is, as might be expected, conflicting. Persons taking an iconoclastic position often attract disproportionate attention, and a sufficiently epigrammatic attack

on a tradition associated with past difficulties will gain at least transient sympathy. The committee is not convinced that in public opinion generally there has been any real impairment of the appreciation of mathematics as a fundamental high school subject. It must be remembered of course that in the general democratic evolutoin of high school education, many new curricula have grown up. It need not be insisted that for all of these, mathematics is of the same relative importance. On the other hand, there is real danger that depreciation of mathematics by persons supposed to be expert in matters of secondary education may, unless vigorously met, exert an unfavorable and undue influence on public opinion. It behooves those who believe in high school mathematics to state their case, not trusting merely to the protection of educational inertia. As evidence that there is still substantial appreciation of mathematics on the part of the more intelligent public, reference may be made to an article by Professor Hancock in *School and Society,* June 19, 1915.

The Test of Utility.—The more familiar criticisms of high school mathematics deny the traditional claims as to its value, insist that its content shall be based mainly on utility, and urge the equal or superior value of such recently or partially developed subjects as general science, elementary civics, etc., etc. The argument for utility profits by the increasing public interest in vocational education. The present committee has no direct concern with vocational education as such. We are not unmindful of the long-existing need of the community in this direction, and we gladly recognize the advantage to the community and to the children of bridging the gap between general education and a self-supporting occupation. On the other hand, it would be foolish in an access of enthusiasm for the new to discard what is good in the old. We contend that preparation for life is something larger and finer than preparation for any mere vocation, that there is far more danger under American conditions of too little general education than too much, and that the criterion of utility, narrowly interpreted, is a wholly inadequate test. As a matter of course the boy or girl should devote the invaluable years of adolescence to activities ultimately useful. Such immediately marketable acquisitions as typewriting and stenography make a sufficiently direct appeal to need no

reinforcement with the average boy or girl. But the value and efficiency of life ten or twenty years later will not depend merely upon these minor factors. The youth need not be less fitted for self-support because his education has provided him with broad and inspiring ideas, nor is such fitness the sole end of life. Whatever one's occupation, it will be fundamentally and permanently important to have acquired in youth the habit of exact, orderly and logical thinking, which, if the experience of many centuries of teaching can be trusted, is best acquired by most high-school students in mathematics well taught.

Educational Values.—Our mission is not merely to occupy our pupils' time, nor to make them efficient but unintelligent artisans; it is to educate them to take their place as good citizens of a free community. It seems to us utterly inadequate to propose the substitution of other subjects "severely pursued." The non-mathematical subjects are too different in kind to serve as proper equivalents. A pupil may doubtless work hard in civics, or chemistry, or composition, but none of these can, for the average boy or girl, fully replace geometry or algebra in its emphasis on form or in the complete validity of its logic.

Some of our psychological authorities are fond of asserting that mathematics "does not function" in the life of the child. We can hardly answer this better than in the words of G. St. L. Carson interesting Essays on Mathematical Education.

"The purpose of teaching natural science is to develop in combination the powers of observation and speculation; to train the pupil to use his senses and from the material which they afford him, to frame hypotheses which accord with the material as nearly as may be possible. The purpose of teaching mathematics is to enable him to develop the consequences of these hypotheses, to test their consistence, and to reduce them to the minimum of pure assumption. I say to train him in these things, but it were perhaps better to speak of setting them before him as ideals for which he must strive in his dealings with things as he finds them. A man who has in his mind this chain of processes, observation, speculation, proof of consistence in speculation, rejection of redundant speculation, and finally the erection of deductions on this foundation, is in possession of an intellectual creation which, in beauty alone, is worthy to rank with

the creations of poetry, music, or art; and beyond this, it is a possession which, in so far as it guides his life, will make of him a more efficient laborer and a better citizen."

"The whole world is going through a transformation, due in part to scientific and mechanical inventions and in part to the growth of separate nations, each with its own methods and ideals, of which no man can see the outcome. Our function, the function of all teachers, is to produce men and women competent to appreciate these changes and to take their part in guiding them so far as may be possible. Mathematical thought is one fundamental equipment for this purpose, but mathematical teaching has not hitherto been devoted to it, because the need has but recently arisen. But now that it has arisen and is appreciated, we must meet it or sink, and sink deservedly. Neither the arid formalism of older days nor—I say it in no spirit of disrespect—the workshop reckoning introduced of late will save us. The only hope lies in grasping that inner spirit of mathematics which has in recent years simplified and co-ordinated the whole structure of mathematical thought, and in relating this spirit to the complex entities and laws of modern civilization."

The Disciplinary Value of Mathematics.—The real development of mankind lies in the growth of voluntary attention, which is not passively attracted, but turns actively to that which is important, significant and valuable in itself. No one is born with such a power. It has to be trained and educated. This great function of education is too much neglected. As a reaction against a rigid, empty, mechanical instruction, there swept over the country a wave of electivism which was meant to bring the blessings of freedom, but which did bring primarily a destruction of self-discipline.

The study of mathematics is valuable in helping to form habits of logical memory. We are told that the memory cannot be trained, for there is not one memory, but many; that everything learned is a new memory. On the other hand, we learn that memory is a property of nerve substance, the tendency of a neurone to retain an impression. The neurones in different parts of the brain must obey the same laws; so while there are many specific memories, there are methods which are common to all memory work. Wherever the habits of memory are

formed, they are formed in obedience to these laws and they are consciously formed. The memory habits formed by the study of algebra and geometry are those from which a logical method of memorizing is developed. While the memory is a special memory, its technique is general and this technique should be learned in connection with the best possible material.

Among the habits related to the content of the subject are the habits of economy of thought and of clearness, brevity, and precision in expression. Mathematics is the shorthand language of abstract thought. In learning how to interpret and master its symbols, habits are acquired similar to other language habits, but here the language habits are associated with the symbols which give the greatest economy in thinking and whose use is universal. The world's work requires the constant mastery of symbols, and thinkers are more and more trying to express laws and conclusions as mathematical formulas. "It is only in the least desired occupations that men work entirely with actual things."

While general power is not necessarily gained by the study of mathematics, the student may be so impressed by the perfection of mathematical reasoning, that an ideal is formed which gives a standard by which his thinking in all other subjects is consciously tested. It is generally recognized that "precise thinking is gaining ground over vague theorizing in many scientific and practical activities of the day and elementary mathematics has been and is the model of precise thinking in more difficult fields."*

Various types of thinking are necessary to solve all the problems of life. There is one method for mathematical reasoning, another for the physical science, which differs from that of the biological sciences, while the psychological and sociological sciences have their own special methods of thought. Every high-school pupil should know the rudiments of all these methods; for no one can take the place of the other, and all are needed if we are to make correct judgments and right decisions in all the affairs of daily life.

The habits of orderly accuracy of thought and the methods of persistent concentration in attacking problems have in them

* J. W. A. Young.

elements that are common to habits formed in other ways, and in so far as two subjects have identical elements, the transfer of training from one to the other will be possible. Since mathematics has many applications to other subjects and to life, there are many opportunities for this transfer, but the ability to make it will depend upon the mental attitudes, upon the aims and ideals which have been formed.

Herein lies the greatest value to be derived from the study of elementary algebra and geometry. The ideals of neatness, accuracy and systematic arrangement will influence all other work. All thought expression will be more direct and clear and concise as it follows the models of mathematical thinking. Unconsciously we shall test all our statements by the logical standards of geometry. Careless methods of work will not give satisfaction; that will come only when there is systematic arrangement and careful classification.

These greatest values, like all the greatest things of life, cannot be reduced to exact measurement. A person is bigger than the sum total of all his specific habits by just these immeasurable attitudes and ideals. What the experiments of the psychological laboratory have shown us is the great influence of the mental attitude upon the formation of the various special habits. When there is a definite thing to do with a definite purpose for doing it, the results obtained are far superior to those under other conditions. Just here is where mathematics excels other subjects, for nowhere else can such definite tasks be assigned and such clearly defined ends be given.

Since the formation of these attitudes of mind and ideals of method must be made consciously, much depends upon our insight and ability to present the subject with more inspiration, with a better defined aim, and with a clearer view of its values; so that the transfer of these attitudes and ideals to other subjects will be made possible.

To put the matter more specifically, it may be expected that in the high-school study of algebra pupils can obtain things of fundamental advantage to them in further thinking on any subject, as follows:

1. Direct and careful statement of their own previous knowledge, as to numbers and their relations, and as to measurable

objects. This will in the case of almost every pupil include facts—for example, about inequalities—of which their knowledge was unconscious.

2. Practice in examination of what might be called narrative statements, in the selection of material facts therefrom, and in the expression of those facts in symbolic form. In a great many cases this includes the stripping of irrelevant connotations from a term, and the simplification of the concept denoted by it.

3. The development of an abstract symbolic statement into new forms, and the concrete interpretation of those new forms.

4. The enlargement of the concept of number, the correlation of number with measurable objects, and the idea of continuous number. However vague this idea may be in the pupil's mind, it is expected and generally attained in the study of this early algebra and geometry.

5. Repeated experience of logical argument.

6. Experience in taking the results of one argument as the data of another, the result of the second as the data for a third, and so on.

7. Appreciation of the advantage of technical terms, and of the advantage of a non-redundant definition. The nature of a definition (genus and specific difference).

8. A great deal of practice in exact statement, and in the appreciation of exact statement, of simple and novel facts.

Mathematics as a Test of Aptitude.—The committee is of opinion that a primary purpose of general education is to afford the student a test of his aptitude. It is a fundamental principle of our democratic society that career shall depend upon capacity and aptitude, not merely upon external circumstances. Fitness for important fields of professional activity—engineering and the exact sciences—is determinable to a considerable extent in the high school by aptitude for mathematics.

It seems to us hardly possible to determine such aptitude—or inaptitude—earlier than about the fifteenth year. Every boy who is to graduate in a general high school course should have there at least an introductory course in algebra and geometry, designed largely as a test of the advisability of further study in that field.

College Admission.—College education, however wide its pos-

sible range, should certainly imply a considerable emphasis on that habit of exact, orderly and logical thinking which we have attributed to high school mathematics well taught.

It does not appear that any considerable number of young people who ought to go to college are at present deterred from so doing by entrance requirements in mathematics. It seems to us that it would be entirely unsafe for a college to accept any other present substitute for a moderate requirement in algebra and geometry.

As we are not aware of any present tendency in this direction on the part of colleges, or any demand for it on the part of schools, in New England, more extended discussion of the question seems needless.

The School Curriculum.—The question of time is naturally fundamental in all curricula. If time were abundant, even the critics of mathematics might be mild; with conditions as they are, it is only fair that the teachers of mathematics should use what time they are assigned with the utmost economy and should be ready to meet competitive claims by justificatory argument as to the relative value of particular topics.

European Curricula.—In a recent publication* of the United States Bureau of Education a comparative study is made of the mathematical curricula by years in the United States and foreign countries.

Beginning with the eighth school year—corresponding to the year next preceding our high schools—

"It is customary in all of the European countries to teach algebra and geometry simultaneously. During the eighth school year the time is about evenly divided between these two subjects, from two to three hours a week being devoted to each. An attempt is not made to fuse the subjects, but the interrelations between them are kept constantly in mind, and the pupil is not permitted to forget his geometry while studying his algebra, or vice versa. Each subject is considered an instruction unit, but it is used whenever possible as a tool in the study of the other. By the time a European boy has completed his eighth school year, he is at least a full year in advance of the American boy in his knowledge of mathematics."

* Curricula in Mathematics, Bulletin, 1914, No. 45, by J. C. Brown.

"In all European schools both algebra and geometry are taught during the ninth school year. In most of the countries the time is divided evenly between these two subjects, from two to three hours a week being devoted to each. Here, as in the preceding year, the subjects supplement each other, and no attempt is made to fuse them. The relations between the two subjects are emphasized much more extensively abroad than in the United States. In most of the schools of Europe the distinction between plane and solid geometry is less marked than in the schools of the United States. This is, in part at least, due to the fact that models and drawings are very extensively used abroad. During the ninth school year the work in algebra in most of the European schools is but slightly more advanced than in the best schools of the United States. In some of the schools of Austria, France, Denmark, Holland and Hungary the course is somewhat more advanced than in the United States. In these schools logarithms, proportion, and quadratics are studied."

"The tenth school year is the second year of the secondary school in the United States. In most of the schools the entire year is devoted to the study of plane geometry. In some of the schools half of the year is devoted to the study of algebra and the other half to the study of plane geometry.

"In the schools of Europe the mathematics of the tenth school year is distinctly in advance of that in the United States. The difference in the mathematical courses in the two countries is most marked in the subjects of geometry, trigonometry and drawing."

"The eleventh school year is the third year of the secondary school of the United States. In most of the schools the course in mathematics includes a half-year of algebra and a half-year of solid geometry. In a few of the schools, the first half-year is devoted to the study of solid geometry and the last half to the study of trigonometry."

"When a European boy has completed the eleventh school year, if he has elected the scientific course, he has studied more mathematics than is offered in any except a very few of the most progressive secondary schools in the United States."

"The twelfth school year is the last year of the secondary school in the United States. In many of the schools the first

half of the year is devoted to the study of solid geometry and the last half to trigonometry or to business arithmetic. In some of the schools the first half of the year is devoted to the study of plane trigonometry and the last half to the study of college algebra.

"In practically all of the European countries the twelfth school year begins or ends with a comprehensive review of the mathematics of the preceding years. Differential and integral calculus are offered in the schools of Austria, Belgium, Denmark, France, Sweden, Switzerland, Russia, Germany and Roumania.

"The relations between algebra and geometry are especially emphasized in France and the relations between mathematics and physics receive special emphasis in Germany, Holland and Switzerland.

"When a European boy has completed his twelfth school year he has had the opportunity of studying more mathematics than is offered in any of the secondary schools of the United States. He has had more practice in applying his mathematics in physics, cosmography, and mathematical geography than is the case with the American boy. The simultaneous study of several mathematical subjects results in a more complete mastery of each. He sees the unity of mathematics in a way that is seldom true with the American boy. He can use his arithmetic and algebra in the solution of geometrical problems and his arithmetic and geometry in the solution of algebraic problems much better than the average American boy. He has some knowledge of analytic geometry and of the infinitesimal calculus. The frequent drills and reviews so common in European schools have furnished him with a larger number of mathematical facts and formulas that he can use more readily than his American brother. Mathematics to him is an interesting and a fruitful subject, because he has learned to appreciate something of its deeper significance."

"Abundant provision is made for daily drill in mathematics. The educator of Europe realizes that this daily drill is absolutely necessary in order to give the pupil a real mastery of number facts and relations. A little smattering of the subject will not suffice. The pupil is expected to know thoroughly certain facts and principles and to this end daily drill is provided. No small

part of the thoroughness in detail, which is so characteristic of most of the schools of Europe, may be traced to this drill. The American pupil has some information on a great variety of topics, but much of his knowledge is vague and indefinite, rather than clear-cut notions about definite things.

"Everywhere algebra is introduced earlier than in the United States. In certain of the German schools some work in algebra is introduced during the sixth school year, and in no country, except the United States, is this introductory work posponed later than the seventh school year.

"Some instruction in constructional, observational, or intuitive geometry is always offered during the sixth, seventh and eighth school years. Much emphasis is placed upon estimates and constructions.

"Everywhere the attempt is being made to find genuine applications of mathematics that are really within the experience of the pupil and to link the subject of mathematics as closely as possible with the activities of real life. Drawing and physics are frequently taught by the same teacher, and the correlation between these subjects is found to be to the advantage of each.

"European school men believe that a course in mathematics should be planned by those who know some mathematics rather than by educators who are practically ignorant of the subject. The reports do not indicate that the schools of Europe are hearing a demand for weak algebra and anæmic geometry, or even for no work in these subjects. If any pressure of this sort exists, it has hitherto produced no modification of the course of study."

It might seem out of place for the present committee, as an interested party, to formulate its own views as to the time requirements of mathematics as against other subjects. The cumulative evidence of the experience of other countries is however certainly significant and impressive. It may also fairly be added that there are two important distinctions between mathematics and some of its newer competitors. In the first place it will be not a few decades before they will be worked into thoroughly teachable condition—involving as this does not merely the discovery, correlation and selection of facts, but the preparation of both teacher and textbooks. In the second place, mathe-

matics must be learned in the school or not at all; general science, civics, etc., permeate our modern environment.

Improvement of Present Conditions.—None of the questions so far discussed seem to us comparable in real importance with those as to possible improvement. People who disagree as to value or utility of mathematics are not easily brought even to a common basis for argument.

But there is continual need of improvement and if we neglect any effort in that direction we shall have failed in a vital obligation.

The best possible way to meet this increasing competition of other subjects is to improve our own both as to content and methods. Some of the criticism aimed at us connects itself with conditions long since outgrown in the best schools. The committee, like the Association, must stand for the best in mathematical education.

IV. Conclusions and Recommendations.

High School Requirements.—As shown in Mr. Morse's report, it is no longer generally true that high-school graduation depends on ability in, or the completion of, a substantial mathematical requirement. We, therefore, offer no specific recommendation on this point beyond that embodied under curriculum, below.

College Entrance Requirements in mathematics seem to us not unreasonable in amount for candidates in general. It seems reasonable, and is a fact, that the boy or girl who might well attend college in spite of inability to meet such a requirement has opportunities to do so, though less freely in this than in other parts of the country.

Curriculum.—In view of the evidence in the country at large that above the sixth grade public-school education is likely in the future to be organized on the basis of a junior high school and a senior high school, each three years in length, we have preferred to base the following recommendations on that plan. Such modifications as would be needful to fit present conditions may be inferred. In six-year high schools, the first three grades (junior high school) should include

(*a*) the essentials of arithmetic, omitting the less important denominate numbers, and all intricate matters of commercial arithmetic, which lie outside the possible experience even of teachers. The simple arithmetic of the home, of the farm, of industry, of ordinary banking, of simple investments and similar topics should be emphasized. This should include a very thorough drill in computation, easy methods of approximation, percentage and its varied applications;

(*b*) the elements of algebra as a condensed notation for the processes already studied in arithmetic, with special emphasis on important formulas and the solution of easy equations including quadratics; also the solution of simple problems;

(*c*) in geometry, a considerable amount of informal geometry —constructive or intuitional—with varied problems in mensuration; the beginnings of geometrical demonstration. Statistical graphs and graphs of simple equations should have been used in connection with (*a*) and (*b*).

To this extent mathematics should be a required subject. Enough should have been taken to test aptitude for further similar study, or for college preparation.

In the *senior high school*—tenth, eleventh and twelfth grades —instruction should be offered

(*a*) *in arithmetic,* the more advanced or more specialized vocational topics; the use of logarithms and the slide-rule;

(*b*) *in algebra,* a second course, differing from the earlier one in being more formal and more logical, including a thorough treatment of quadratic equations, due attention to such matters as radicals, exponents, factoring, binominal theorem, etc., and the solution of more difficult problems;

(*c*) *in geometry,* the more important theorems of plane and solid geometry, additional work in mensuration; the trigonometric solution of the right triangle;

(*d*) *in trigonometry,* the theory and solution of the plane triangle, with varied applications.

In (*b*), (*c*) and (*d*) the idea of function—*i. e.,* of related variables—should be introduced; also if possible, some simple illustrations of the derivative—*e. g.,* slope and speed.

How much of this program would be taken by the individual student would depend upon circumstances, but *trigonometry,* on

account of its simplicity and interest, should be generally attractive.

How far the several disciplines should be combined in composite courses seems to us a matter still awaiting experimental determination. We are inclined to favor the general principle of parallel rather than blended courses in algebra and geometry, but some overlapping of each by the other is desirable, as in the use of graphs in algebra and of algebraic methods in geometry.

In *four-year high schools* much will naturally depend on the possibilities of the grammar school preparation, but in our judgment the aims should conform in a general way to the above for corresponding grades. At the end of the first year in the four-year high school, the student should have had a variety of elementary work in both the solution of algebraic equations and in observational geometry and mensuration, with a few simple proofs. In the second year he should be enabled to choose between elementary trigonometry, higher arithmetic and college preparatory work in algebra and geometry.

(*A*) At the end of the first year the student should have cogered substantially the ground indicated as desirable for junior high school work. Omissions, if any are necessary, should be rather of application and practice than of important topics. The committee is of the opinion that to this extent mathematics should be required in the four-year high schools.

(*B*) In the second and succeeding years the pupil should have the opportunity to continue and complete the course of study in mathematics as outlined above for the senior high school.

Preparation of Teachers.—It is the belief of the committee that the preparation of teachers of mathematics in the senior high schools (above the ninth grade) should include the completion of not less than three years of college mathematics, comprising besides the ordinary trigonometry, analytic geometry, calculus with applications, and practice in geometrical drawing work in one or more of the following fields: higher algebra, modern geometry, mechanics, or applied mathematics. In addition, some knowledge of the history and fundamental concepts of mathematics, and some professional training in the teaching of mathematics are desirable.

Teachers of mathematics in the junior high school (seventh,

eighth and ninth grades) should have had instruction in plane and solid geometry, trigonometry, and college algebra (including graphs), or an equivalent with much attention to concrete applications, *i. e.*, the ordinary first-year college mathematics. Preparation to this extent should be regularly provided in the normal schools, as well as instruction in the history and teaching of arithmetic, algebra and geometry.

Procedure.—Recognizing that the changes proposed above are much too far-reaching to be dealt with independently by this Association, the committee presents them as a basis for further study and discussion in co-operation with similar bodies in other parts of the country and with the Mathematical Association of America. It is respectfully recommended that the present committee be discharged and that the council be requested to open negotiations with such other bodies with a view to the organization of a national joint committee, or to such other action as may seem best calculated to secure thorough and adequate consideration of the above recommendations on the part of teachers, administrative officers, and colleges.

G. W. Evans, *Charlestown High School;*
F. C. Ferry, *Williams College;*
A. V. Galbraith, *Middlesex School;*
F. P. Morse, *Revere High School;*
C. D. Meserve, *Newton High School;*
Harriet R. Pierce, *Worcester High School;*
Sophia E. Smith, *Mt. Holyoke College;*
H. W. Tyler, Chairman,
Mass. Institute of Technology.

Appendix I.

An Investigation with Regard to Algebra and Geometry in the High Schools of Massachusetts. By F. P. Morse.

In April, 1915, a questionnaire was sent to the larger high schools of the state outside of Boston. The principals of these schools were asked five questions:

1. Is algebra required for graduation in your school?
2. Is geometry required for graduation?

3. In what years are these subjects offered?

4. How many times per week?

5. Do you think that algebra and geometry should be required for admission to college?

Ninety-seven copies of the questionnaire were sent out and eighty-seven replies received. A summary of the replies is given below.

Is algebra required for graduation? Yes 16, No 65.

Five schools require the subject but allow the pupil to be graduated even if it is not satisfactorily completed. One school requires only sixteen weeks of simple equations and problems. It is evident therefore that in 70 of the 87 schools from which replies were received it is possible for a pupil to receive a diploma without completing a course in algebra.

Forty-seven of the 65 replies which are tabulated under number one indicate that in the schools from which they come there is no requirement in algebra outside of the college course, while the other 18 schools require algebra in several courses, or in most courses.

The replies to question No. 2 give the following results:

Is geometry required for graduation? Yes 8, No 79.

Seventeen of the 79 schools tabulated under *No* require geometry in several courses.

Under question 3, of 81 replies that could be tabulated, a large majority reported that algebra is begun in the first year of high school and geometry in the second. In fact only 3 schools reported that algebra is begun in the second year. Two schools have both subjects in each of the first two years.

The last question brought replies showing a variety of opinions. Fifty-three of those replying said yes. Ten said no. Six did not think these subjects should be required for all courses. Two say they should be required only of those who will make subsequent use of them. Seven would make a difference between boys and girls, either not requiring the subjects at all of the girls, or requiring less than of the boys. Four would require one subject, not both. One reply says: "It all depends on the character of the college and the student's plans." Another: "They seem to me a vestigial survival." Another: "Not necessarily. But anyone thinking of going to college ought to be sure he has brains enough to handle them."

It was not possible to tabulate the replies to the fourth question, but it would appear that 4 or 5 periods per week are most commonly given to the first courses in algebra and geometry.

APPENDIX II

Preparation of the Teachers of Mathematics in the Secondary Schools of Massachusetts

In a report published by the Kentucky Department of Education in 1909 it is stated that a certain young woman planned to make her entire preparation for the teaching of German in "one of the leading high schools" in Kentucky by devoting to the study of that language six weeks at a summer school. The authority for this statement "insists" that 90 per cent. of the modern language teachers in the American schools "do not perform better work than this lady did in first-year German"; one infers that their preparation is thought to be no more thoroughly established than hers.

It is reported that in 1908 only 32.2 per cent. of the 4,668 secondary school teachers in the State of New York were college graduates.

In that portion of the American Report of the International Commission on the Teaching of Mathematics, prepared by Committee III, page 81, one reads that "we must remember that the majority of those teaching mathematics in our high schools are prepared merely on the side of subject matter, while it is safe to say that in many of our smaller high schools algebra and geometry are taught by those who are not familiar with any mathematics beyond these same subjects." The American Commissioners of the same International Commission make the report of the same Committee III their authority for the statement concerning the secondary schools (page 35 of Bulletin, 1912, No. 14) that "The average newly appointed teacher of mathematics is a college graduate who has had only about one year's work (from 90 to 180 class hours) of mathematics beyond the work of the school in which he teaches."

While one must accept the statements of such authorities as true of the country as a whole, the experience of some of those

having to do with teachers of mathematics in the secondary schools of Massachusetts made them seem inapplicable to this particular state. To determine whether one's favorable impression of the preparation of this body of teachers was warranted appeared an interesting task. This problem looked still more inviting when the State Commissioner of Education in Massachusetts, who could be expected to speak with authority, wrote in his letter of July 20, 1914, to the Committee on the Status of Mathematics in Secondary Schools as follows: "In most small high schools today it will be found that the teacher with least special preparation for his work is usually teaching algebra." The commissioner did not except the schools lying within his jurisdiction and one felt that the statement was intended to apply particularly to them. Inasmuch as it could not be learned that anyone had made a study of the question of the preparation of the teachers of secondary mathematics in Massachusetts as a whole, the task was undertaken.

There were sent in April, 1915, and again in January, 1916, to the principals of the 264 high schools listed by the Massachusetts Board of Education letters with enclosed postal cards of questions. The principals were requested to distribute these cards to their teachers of mathematics. The questions proposed to the teachers in the first questionnaire were as follows:

1. Do you teach mathematics only?
2. If other subjects also, please name them:
3. Are you a college graduate?
4. Please underscore the subjects studied by you in college or under instruction elsewhere: College algebra, solid geometry, trigonometry, analytic geometry, differential calculus, integral calculus.
5. Have you had instruction in any of these since graduation from college?
6. How many years have you taught? How many where you are now?
7. Are you a man or a woman?

The questions contained in the second questionnaire were these:

1. What college degrees have you?
2. Are you a graduate of a normal school?

3. Have you taken courses in education? Pedagogy? Teaching of mathematics? History of mathematics? How many semester courses in these subjects? How highly do you value this work in your own case?
4. What subjects in mathematics have you studied beyond trigonometry, analytic geometry, and calculus?
5. How long have you taught?
6. Do you teach mathematics only?
7. Are you a woman?

Replies to the first questionnaire came from 398 teachers, 240 men and 158 women, representing 245 of the 264 schools; and replies to the second came from 369 teachers representing 217 schools. Although some changes in personnel must have taken place between April, 1915, and January, 1916, the teachers who answered the two questionnaires are in general the same, and the replies have been considered as coming from the same body of teachers in both cases.

Of these 398 teachers, 40 per cent. teach mathematics only, 36 per cent. teach mathematics and one other subject (which in half the cases is science), 19 per cent. teach mathematics and two other subjects, 4 per cent. teach mathematics and three other subjects, four teach mathematics and four other subjects, while one teacher combines mathematics with science, English, French, civics, agriculture, history, and athletics!

93 per cent. of these teachers are college graduates, which leaves only 27 who lack the bachelor's degree. These 27 teachers have in several instances had two or three years of college study; they have taught in the average 23 years each; accordingly they passed the college age before attendance at college became a very common thing. Of those graduated from college, 81 per cent. have the degree of A.B., 14 per cent. the degree of S.B., three per cent. the degree of Ph.B. and two per cent. the degree of Litt.B. 13 per cent. of these college graduates have a second degree, which in eight cases out of nine is the A.M., and in a few instances is an engineering degree. One teacher has three degrees,—S.B., A.M., and M.D.; one has the degree of LL.B.; and none has the degree of Ph.D. or Sc.D.

10 of the 398 teachers state that they have studied in college or under instruction elsewhere none of the subjects listed; 97

per cent. of these 398 teachers of secondary mathematics have studied in college or under instruction elsewhere at least as far as through college algebra, solid geometry, and trigonometry; 71 per cent. of the 398 have studied thus through these subjects and analytic geometry; 54 per cent. continued through differential calculus, and 48 per cent. through integral calculus. Thus it is found that less than three per cent. of all these teachers failed to complete at least a year of college mathematics, while nearly half of them carried the study of the subject through three or more college years. The 10 teachers who have done no college mathematics under instruction average to have taught more than 15 years each. Five of them are college graduates and the five non-graduates average to have taught about 23 years each. Only two of the 10 teach mathematics only, and those two have taught 22 and 26 years respectively.

27 per cent. of this entire number of teachers have had instruction in mathematics since leaving college. This means that more than a quarter of these teachers have interested themselves in their subject to the extent of taking courses in it since their undergraduate days.

The subjects which these teachers have studied beyond trigonometry, analytic geometry, and calculus are of interest. Allowing them to be the judge as to the subjects that should be regarded as lying "beyond" the calculus, and using in general their own characterizations of these subjects, it is found that 29 per cent. of the whole body of teachers have studied such subjects. Of these teachers comprising the 29 per cent., one third have studied mechanics; one fourth have studied projective geometry; one fourth have studied differential equations; 21 per cent. have studied theory of equations; 13 per cent., modern analytic geometry; 9 per cent., descriptive geometry; 8 per cent., theory of functions; 7 per cent., mathematical physics; five per cent., determinants; four per cent., Fourier's series; four per cent., theory of least squares; while some two or three have studied each of eight or nine other mathematical subjects regarded as of advanced character.

While 93 per cent. of these 398 teachers are college graduates, only about 8 per cent. are normal school graduates. Half of the graduates of the normal schools are college graduates also.

The four per cent. who were graduated from the normal school and have no college degree average to have taught 23 years each. Apparently the time is past when the normal school course is regarded as affording sufficient preparation for the teaching of mathematics in a secondary school in Massachusetts.

The replies to the second questionnaire enable one to determine how much of professional training this great body of teachers of mathematics has had through the study of pedagogical subjects. It is found that 57 per cent. have taken courses in education; 44 per cent. have taken courses in pedagogy; 29 per cent. have taken courses in the teaching of mathematics; and 15 per cent. have taken courses in the history of mathematics. The teachers who have taken these courses average to have pursued the study of them to the extent of about three semester courses. 30 per cent. of the teachers who have taken these courses report them to have been of "great" value, 19 per cent. regard them of "some" value, 19 per cent. regard them of "very little" value, four per cent. think them valueless, and 31 per cent. make no reply to the question. One teacher, an A.B. of twenty years of teaching experience, states that these subjects are of "blamed little" worth, but admits that he has not studied them. One "would have felt lost" without this training; another declares that he values them "little as actual assistance in teaching, but highly as a gain in power and confidence." The normal school graduates appear to esteem this work no more highly than do those who have not attended a normal school, although a much larger percentage of the former have taken such courses.

The 398 teachers report an average teaching experience of twelve years. 22 have taught only a single year, 22 have taught two years, 17 three years, 33 four years, 25 five years, 73 from 6 to 9 years, 69 from 10 to 14 years, 57 from 15 to 19 years, 64 from 20 to 29 years, 11 from 30 to 39 years, and four for 40 years or more.

As showing the relative rate of change of positions, the reports show that these 398 teachers average to have remained continuously in their present teaching positions for between six and seven years. 49 of these teachers have occupied their present places continuously more than 15 years, 22 have held

them for more than 20 years, five have done so for more than 30 years, and one has occupied the same chair for 40 years.

Inasmuch as the preparation of those teachers who devote all their time to mathematics only is of special importance in this investigation, their returns have been considered separately with the following results. 160 teachers, of whom 92 are men and 68 women, teach mathematics only. Of these, 94 per cent. are college graduates; and the nine non-graduates, one man and eight women, average to have taught 31 years apiece. Only two of these 160 teachers fail to have studied in college or under instruction elsewhere through trigonometry and those two have taught 22 and 26 years respectively. 81 per cent. of these teachers have studied thus through analytic geometry, 65 per cent. through differential calculus, and 58 per cent. through integral calculus. 32 per cent. of these 160 teachers of mathematics only have had instruction in mathematics since leaving college,—as compared with 27 per cent. of the entire body of 398 teachers. These 160 teachers average to have taught fifteen years and to have held their present positions eight years. Nine of them, with an average of 17 years of teaching experience, have taught in only a single school. In every respect these teachers of a single subject show better reports than do the teachers of two or more subjects.

In order to determine the relative preparation of the teachers of mathematics in the small schools, 81 teachers from 81 of the smallest high schools in the state were selected. Their answers show that 59 of them are men and 22 women; that none of them teaches mathematics only, and that 76, or about 94 per cent., graduated from college; that 79 of the 81 have studied through trigonometry, 62 per cent. through analytic geometry, 48 per cent. through differential calculus, and 42 per cent. through integral calculus; and that 11 per cent. have studied mathematics since leaving college. Even in these smallest high schools it is hard to believe in the face of these facts that "the teacher with least special preparation for his task is usually teaching algebra." These 81 teachers average to have taught only seven years, and to have continued in their present positions only two years. Only four of them have taught as many as 20 years. These are in general the young teachers who make their

beginning in the very small high schools and rapidly advance to better places in larger schools.

In comparing the returns of the 240 men with those of the 158 women, it is found that 38 per cent. of the men and 43 per cent. of the women teach mathematics only, and that 96 per cent. of the men and 87 per cent. of the women are college graduates. 97 per cent. of the men and the same percentage of the women have studied through freshman mathematics, 65 per cent. of the men and 69 per cent. of the women have continued through analytic geometry, and 46 per cent. of the men and 50 per cent. of the women have carried their studies in college through the integral calculus or still further. 24 per cent. of the men and 26 per cent. of the women have had instruction in mathematics since leaving college. In general, then, the women show better preparation for their work than the men. Striking differences between the figures for the men and those for the women appear when the length of the period of teaching is considered. Thus, 9 per cent. of the men and 19 per cent. of the women have taught only one or two years; 10 per cent. of the men and only two per cent. of the women are now engaged in their sixth year of teaching; and 16 per cent. of the men and 24 per cent. of the women have taught more than 20 years. It appears, therefore, that the women who forsake a teacher's life do so ordinarily after only one, two, or three years of it, while the men who give up secondary teaching usually do so after at least ten years of that work have been completed.

The men change their teaching positions more frequently than the women. 55 per cent. of the men and only 43 per cent. of the women have remained in their present positions less than four years. Of the men only 29 per cent. have continued in their present places more than six years, while 48 per cent. of the women have so continued. The percentages of teachers who have taught in only one school and have served more than two years there are four for the men and ten for the women.

It seems to have been established that the statements found in many articles, particularly those of the non-mathematical officers of education, concerning the great lack of preparation of the teachers of mathematics in the secondary schools are not applicable to the state of Massachusetts. When 93 per cent.

of these teachers are college graduates, about half of the entire number have studied through the integral calculus in college, more than a quarter of them have had instruction in mathematics since leaving college, and more than half have had professional training in the teaching of their subject,—the average newly appointed teacher of mathematics in the Massachusetts high schools is worthy of a far better characterization than that which has been quoted above from the American Commissioners. Undoubtedly the State of Massachusetts is far more fortunate in this regard than most of the other states; but may one not hope that improvement is showing itself everywhere and that those who speak only after careful investigation will soon find better things to say of the preparation of the teachers of mathematics in the secondary schools of the entire country!

The ideal preparation, even when as moderately characterized as is the case in the report of the present committee, may be long deferred; but the reports of the 398 Massachusetts teachers of algebra and geometry give ground for hope.

Appendix III.

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